Internal Report

Gravitational waves from a binary system: A detailed analysis of orbital decay

LEE Sze Him

Department of Physics, The Chinese University of Hong Kong, Hong Kong, China

Samuel LAI

Department of Physics, UCLA, USA

(Dated: September 6, 2017)

Abstract

The slow decay of the orbits of binary systems provide indirect evidence for gravitational waves. In this Report, we derive a formula for the energy lost to gravitational waves by a general binary system with masses M_1 and M_2, orbital period T, and orbital eccentricity e, using the quadrupole formula. The result is expressed in terms of the reference case of equal masses moving in circular orbits with the same period T, multiplied by two independent correction factors, respectively to account for the unequal masses and the eccentricity. The latter, when written suitably as a power series in e, turns out to terminate. The result is consistent with the observed rate of decay of the binary pulsar system PSR 1913+16. The eccentricity effect is large: a factor of $\sim$10 for moderate values of e.

Two approaches were used to derive the result. The first approach, in the time domain, integrates the power expression (involving the square of the third time derivative of the quadrupole moment) over the period. This method shows clearly why the result is a polynomial in e.

The second approach analyzes the quadrupole moment in terms of harmonics. Quadrupole radiation goes as ω^6, so the nth harmonic contributes with a factor n^6. It was found that the mean n increases rapidly with e, with a value of 8.3 for the eccentricity of PSR 1913+16 (e 0.62), compared to $n = 2$ for a circular orbit. The dominance of the high harmonics provide a physical understanding for the large eccentricity factor.

Contents

I. INTRODUCTION

A. Gravitational waves

In February 2016, LIGO (Laser Interferometer Gravitational-Wave Observatory) announced that a gravitational-wave signal was observed, which agreed with the waveform predicted by general relativity for the in-spiral and merger of two black holes [1]. Four months later, LIGO announced the second detection of a pair of coalescing black holes [2]. The discovery has important implications including direct detection of gravitational waves, explicit observation of black holes and verification of general relativity.

But before LIGO, there was already indirect evidence for gravitational waves. In 1974, Hulse and Taylor discovered the binary pulsar system PSR 1913+16. The most important feature was an observed slow decay of the orbit [3], attributed to the loss of energy to gravitational waves [3]. The observed rate of decay was found to be consistent with the prediction of general relativity based on the radiation of gravitational waves, which established the existence of gravitational waves indirectly. Hulse and Taylor later shared the Nobel Prize in Physics "for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation" [5].

B. Hulse–Taylor pulsar

PSR 1913+16 has a pulsar period of 59 ms and is 21 000 lt-sec away from the Earth [3]. Pulse arrival times showed an anomaly of about 3 s, repeating every 7.75 hr. This means the distance from us is varying be 3 lt-sec; in other words the pulsar is moving in an orbit with orbital diameter projected along the line of sight of about 3 lt-sec and orbital period $T = 7.75$ hr [4]. The observations also indicated that the pulsar was orbiting with another unseen star (the companion, in fact a neutron star [4]) about their common center of mass (CM).

The orbit of the pulsar must be an ellipse; denote the semi-major axis as a_1 and the eccentricity as e. The distance of 3 lt-sec is related to the projection of $2a_1$ onto the line of sight. Detailed analysis of the anomaly gives the following orbit parameters [3]: $a_1 = 3.24$ lt-sec $= 9.7 \times 10^8$ m, $e = 0.617$, $T = 7.75$ hr $= 2.79 \times 10^4$ s. The companion is in an orbit with semimajor axis $a_2 = 3.26$ lt-sec, and of course the same eccentricity [6]. The masses were determined to be $M_1 = 1.42\ M_\odot$, $M_2 = 1.41\ M_\odot$.

The most important observation for our purpose is the rate of orbit decay: $-\dot{T} = 76\ \mu$s/yr.

C. The key result to be explained

The decay rate is characterized by

$$\alpha \;=\; -\frac{\dot{T}}{T} = 2.72 \times 10^{-9}\ \text{yr}^{-1} \tag{1.1}$$

or in terms of the cumulative period shift Δt_c [4]:

$$\Delta t_c \;=\; -(\alpha/2)\, t^2 \tag{1.2}$$

Observation of PSR 1913+16 over decades yields the coefficient [4] $\alpha/2 = 4.3$ s/decade^2 consistent with (1.1). The purpose of this Report is to show that the values of α can be correctly explained by gravitational radiation, using the orbital parameters introduced in Section I B.

D. Quadrupole radiation

In classical electrodynamics, a system of oscillating charges will radiate power in the form of electromagnetic waves. The monopole term is proportional to the total charge Q of the system, which does not vary with time and therefore does not contribute to the radiation. As long as the total charge is non-zero, it is possible to choose an origin such that the electric dipole moment is zero [7]. Even if $Q = 0$, the analogue of the electric dipole moment in the gravitational case is related to the position of the center of mass, which cannot oscillate for an isolated system. Also, the analogue of the magnetic dipole term is related to the angular momentum, which again cannot oscillate. The next term is the electric quadrupole moment, defined as [9]

$$q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\,\rho(x)\,d^3x \tag{1.3}$$

where ρ is the charge density. The power loss due to quadrupole radiation for a system oscillating at a definite frequency ω, transferred to our notation, is

$$P = \frac{1}{360}\frac{1}{4\pi\epsilon_0}\frac{\omega^6}{c^5}\sum_{ij}|q_{ij}|^2 \tag{1.4}$$

where q_{ij} denotes the peak value of the quadrupole moment in its sinusoidal oscillations.

In an exactly analogous way, a system of oscillating masses will radiate power in the form of gravitational waves. The formula for mass quadrupole radiation [7] can be guessed by regarding $\rho(x)$ as the mass density and using the mapping

$$\frac{1}{4\pi\epsilon_0} \longmapsto G \tag{1.5}$$

The correct formula due to general relativity is actually 4 times this result obtained by the naive mapping. Thus, the power radiated by mass quadrupole moment is

$$\boxed{P = \frac{1}{90}\frac{G\omega^6}{c^5}\sum_{ij}|q_{ij}|^2} \tag{1.6}$$

where, in an analogous way, q_{ij} is the peak value of the mass quadrupole moment in its sinusoidal oscillations.

All these formulas are based on the multipole expansion, and keeping only the leading terms. For harmonic motion with an amplitude a, the multipole expansion is in powers of $ka = \omega a/c$, where k is the wavenumber and a is a characteristic size of the source [9]. Also, the typical velocities are $v \sim \omega a$. Thus, the expansion parameter is

$$\frac{\omega a}{c} \sim \frac{v}{c} \tag{1.7}$$

We see that the multipole expansion is equivalent to the relativistic expansion in powers of the factor v/c. In PSR 1913+16, this factor is around 10^{-3}, small enough for the above formulas to be accurate. For merging binaries, however, these arguments are only qualitative.

E. Organization of this Report

The rest of this Report is organized as follows. Section II sets out the reference case with equal masses in a circular orbit. Section III then shows that the general case involves two independent factors, one for the inequality of the two masses, and one for eccentricity, and evaluates the former. The eccentricity effect is then dealt with in Section IV; the key result in (4.14) involves a simple prefactor and a polynomial of degree 4 in e.

The eccentricity effect is surprisingly large (a factor of $\sim$12 for PSR 1913+16, with $e \sim 0.6$), for which a qualitative understanding is sought. One conjecture is that this may be due to high harmonics: quadrupole radiation goes as $\omega^6 = (n\Omega)^6$, Ω being the orbital frequency; thus higher harmonics (absent for a circular orbit) contribute with a weight n^6. This conjecture is examined by evaluating the power radiated by each harmonic, in two independent ways: first using a power-series expansion in e (Section V) and then using numerical integration (Section VI). The former method encounters a subtle and intriguing technical difficulty, but numerical evaluation does confirm that the large eccentricity factor is due to high harmonics.

Concluding remarks are given in Section VII.

II. REFERENCE CASE

A. Definition and framework

The power radiated, P, will be expressed in terms of the analogous quantity P_0 for a reference case with the same orbital period. The latter consists of two equal masses, each M, orbiting around the CM in a circle of radius R. The orbital period is

$$T = 4\pi\sqrt{\frac{R^3}{GM}} \tag{2.1}$$

The general case has masses

$$M_{1,2} = M(1 \pm f) \tag{2.2}$$

moving in elliptical orbits of eccentricity e and the same period. Since by Kepler's third law the period is related to the average semi-major axis $a = (a_1 + a_2)/2$, it follows that the reference case should be chosen with $R = a$.

The task is then divided into two parts: (a) first to evaluate P_0 (this Section), and (b) then to evaluate the ratio P/P_0 (next two Sections). The ratio can only depend on the dimensionless quantities e and f, so

$$\frac{P}{P_0} = F(f, e) \tag{2.3}$$

It will be shown in Section III that the ratio factorizes into the mass effect and the eccentricity effect:

$$\boxed{F(f, e) = F_f(f) \cdot F_e(e)} \tag{2.4}$$

which will then be studied separately.

B. Evaluation of reference case

Let $\Omega = 2\pi/T$ be the angular frequency of orbital motion. Following Young [8], by applying (1.3) to the case of discrete point masses, the q_{xx} term of the quadrupole moment is

$$\begin{aligned} q_{xx} &= \sum_\alpha M_\alpha[3(x^\alpha)^2 - (r^\alpha)^2] \\ &= 2M(3R^2 \cos^2 \Omega t - R^2) \\ &= 3MR^2 \cos \omega t + \text{const} \end{aligned} \tag{2.5}$$

where the index α labels the particles with coordinates $\mathbf{r}^\alpha = (x^\alpha, y^\alpha, z^\alpha)$, $r^\alpha = |\mathbf{r}^\alpha|$, and $\omega = 2\Omega$. Since the two masses are identical, the system repeats itself every half cycle. Therefore, ω is the frequency of the oscillation of the quadrupole moment, and hence of the radiation. In other words, for this reference case with orbital frequency Ω, only the second harmonic radiates. The amplitude of oscillation of q_{xx} is

$$q_{xx} = 3MR^2 \tag{2.6}$$

So

$$q_{xx}^2 = 9M^2R^4 \tag{2.7}$$

It turns out that each of q_{xy}, q_{yx} and q_{yy} gives the same contribution. Therefore,

$$\sum_{ij} |q_{ij}|^2 = 36M^2R^4 \tag{2.8}$$

The power radiated, according to (1.6), is thus

$$P_0 = \frac{1}{90}\frac{G\omega^6}{c^5}\cdot 36M^2R^4$$
$$= \frac{2}{5}\frac{GM^2R^4\omega^6}{c^5} \tag{2.9}$$

Thus, for the reference case with $R = a$, the power radiated is $P = 1.0 \times 10^{22}$ W.

III. FACTORIZATION AND MASS EFFECT

This Section deals with the mass effect, showing that it factorizes from the eccentricity effect as a multiplicative factor F_f. Thus we consider two masses M_1 and M_2, and introduce the dimensionless variable f by (2.2). Let r_α ($\alpha = 1, 2$) be the distance between M_α and the CM [10]. By the definition of the CM,

$$(1{+}f)\, r_1 = (1{-}f)\, r_2$$

and hence

$$r_1 = r\,(1{-}f) \quad , \quad r_2 = r\,(1{+}f)$$

where

$$r = \frac{1}{2}\,(r_1 + r_2)$$

Applying (1.3), the quadrupole moment of a system of masses M_α at positions $\mathbf{r}^\alpha$ is given by

$$q_{ij} = \sum M_\alpha \left[3x_i^\alpha x_j^\alpha - (r_\alpha)^2\delta_{ij}\right]$$

Applied to the present case, we have

$$q_{ij} = M(1{+}f)r_1^2\, h_{ij}(\phi) + M(1{-}f)r_2^2\, h_{ij}(\phi+\pi)$$

where the non-zero elements of $h_{ij}(\phi)$ are

$$h_{xx} = 3\cos^2\phi - 1$$
$$h_{xy} = h_{yx} = 3\sin\phi\cos\phi$$
$$h_{yy} = 3\sin^2\phi - 1$$
$$h_{zz} = -1 \tag{3.1}$$

We see that $h_{ij}(\phi) = h_{ij}(\phi + \pi)$, thus the factors related to ϕ are the same for both terms and can be taken out:

$$\begin{aligned} q_{ij} &= M h_{ij}(\phi) \left[(1+f) r_1^2 + (1-f) r_2^2 \right] \\ &= 2 M h_{ij}(\phi) r^2 (1 - f^2) \end{aligned}$$

The elements of the quadrupole moment q_{ij} are to be regarded as functions of time t, and its time-varying parts lead to radiation. Although the time t and the orbital position ϕ are related in a complicated way which involves the eccentricity: $t = t(e, \phi)$, or $\phi = \phi(e, t)$, the masses enter only through the last factor above. Thus we see that the mass effect factorizes, with a factor

$$\boxed{F_f = (1 - f^2)^2} \tag{3.2}$$

IV. ECCENTRICITY EFFECT

A. Formulation

Given the result of the last Section, it suffices to consider the eccentricity effect for the equal-mass case. In this scenario, the two masses (each M) are equidistant from the CM, and therefore their coordinates are always opposite. Since the quadrupole moment involves quantities quadratic in the coordinates, each mass contributes the same amount. The semi-major axes a_1 and a_2 of the two orbits, and their average value a, are all the same. We therefore consider (and at the end multiply by 2) one mass M orbiting the origin in a Kepler orbit of semi-major axis a. Some basic facts about such a Kepler orbit are given in Appendix B.

Up to some constants that will cancel when we consider the ratio with the reference case, the power P is proportional to

$$P \propto \sum_{ij} \omega^6 |q_{ij}|^2 \tag{4.1}$$

The term $\omega^6 |q_{ij}|^2$ is nothing but the time average of

$$\left(\frac{d^3}{dt^3} q_{ij} \right)^2 \tag{4.2}$$

Hence, and from now on dropping the common factors that will cancel upon comparison with the reference case,

$$
\begin{aligned}
P &= \sum_{ij} \frac{1}{T} \int_0^T \left(\frac{d^3}{dt^3} q_{ij} \right)^2 dt \\
&= \sum_{ij} \frac{1}{T} \int_0^T \left[\frac{d^3}{dt^3} r^2 h_{ij}(\phi) \right]^2 dt
\end{aligned} \tag{4.3}
$$

We define

$$
P_{ij} = \frac{1}{T} \int_0^T \left[\frac{d^3}{dt^3} r^2 h_{ij}(\phi) \right]^2 dt \tag{4.4}
$$

In the Kepler orbit, the distance r has the expression

$$
r = r_0 \, s(e)^{-1} \tag{4.5}
$$

where

$$
s(e) = 1 - e \cos\phi \tag{4.6}
$$

The expression (4.4) is an integral over t, but the integrand is expressed in terms of ϕ. It is therefore necessary to relate these two variables. The relationship is essentially Kepler's second law: the radius vector sweeps out equal areas in equal times, which can be translated into

$$
dt = \frac{r^2}{J} \, d\phi \tag{4.7}
$$

where J is a constant, in effect the conserved angular momentum per unit mass. The expression for P_{ij} becomes

$$
\begin{aligned}
P_{ij} &= \frac{1}{T} \int_0^{2\pi} \left[\left(\frac{J}{r^2} \frac{d}{d\phi} \right)^3 r^2 h_{ij}(\phi) \right]^2 \frac{r^2}{J} \, d\phi \\
&= \frac{J^5}{r_0^6 T} \int_0^{2\pi} \left[\left(s^2 \frac{d}{d\phi} \right)^3 \frac{h_{ij}}{s^2} \right]^2 \frac{1}{s^2} \, d\phi
\end{aligned} \tag{4.8}
$$

B. Evaluation of integrals

The integrals can be evaluated exactly. Note that there is a factor of $(1 - e\cos\phi)^{-2}$ in the integrand. For each differentiation, the factor of $(1 - e\cos\phi)^{-1}$ will move up by one unit.

Hence, after 3 differentiations, there will be a factor of $(1-e\cos\phi)^2$ in the integrand, which gives a finite polynomial in e after the integration. This is a key feature of the calculation, to which we shall return in the next Section.

The details of the algebra are in Appendix C. The expressions for P_{ij} are

$$\begin{aligned} P_{xx} &= \frac{\pi J^5}{r_0^6 T}\left(144+\frac{797}{2}e^2+\frac{565}{16}e^4\right) \\ P_{xy}=P_{yx} &= \frac{\pi J^5}{r_0^6 T}\left(144+\frac{873}{2}e^2+\frac{1107}{16}e^4\right) \\ P_{yy} &= \frac{\pi J^5}{r_0^6 T}\left(144+\frac{953}{2}e^2+\frac{757}{16}e^4\right) \\ P_{zz} &= \frac{\pi J^5}{r_0^6 T}\left(4e^2+e^4\right) \end{aligned} \tag{4.9}$$

The total power P is

$$\begin{aligned} P &= P_{xx}+P_{xy}+P_{yx}+P_{yy}+P_{zz} \\ &= \frac{\pi J^5}{r_0^6 T}(576+1752e^2+222e^4) \\ &= \frac{576\pi J^5}{r_0^6 T}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \end{aligned} \tag{4.10}$$

Note that r_0 and J also depend on e if lengths are referenced to a. The relations are

$$r_0 = a(1-e^2) \tag{4.11}$$

$$J = \sqrt{r_0 K} = \frac{\sqrt{GMa(1-e^2)}}{2} \tag{4.12}$$

Hence, we can write the total power as

$$P = \frac{18\pi(GM)^{5/2}}{a^{7/2}T}(1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{4.13}$$

The values of T, M and a are identical to the reference case of $e=0$. Hence, the ratio P/P_0 is

$$\boxed{F_e(e) = (1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)} \tag{4.14}$$

This expression is positive-definite, as it ought to be. The espression matches with the literature [12].

C. Application to the present case

The eccentricity of PSR 1913+16 is $e = 0.617$, which gives a correction factor of $F_e(e) = 11.84$. In fact, putting all the factors together, we have $P_0 = 1.0 \times 10^{22}$ W, $F_f \approx 1.0$, $F_e = 11.84$, giving $P = 1.2 \times 10^{23}$ W. Hence the predicted orbital decay parameters are $\alpha = 2.7 \times 10^{-9}$ yr^{-1}, $\alpha/2 = 4.3$ s/decade^2, or $-\dot{T} = 76$ μs yr^{-1}, in good agreement with the observed values.

V. HARMONIC ANALYSIS: EXPANSION IN ECCENTRICITY

A. Motivation

In this and the next Sections we examine a conjecture for the reason behind the surprisingly large value of the eccentricity factor ($F_e(e){\sim}12$ for $e{\sim}0.6$), based on two observations. (a) For a circular orbit ($e = 0$), all the coordinates are exactly sinusoidal in time, and the quadrupole moment has only one frequency $\omega = 2\Omega$. However for an eccentric orbit ($e \neq 0$), the coordinates vary periodically but not sinusoidally, and the quadrupole moment has frequencies $\omega = n\Omega$, with various n. (b) The power radiated in any harmonic goes as $\omega^6 \propto n^6$. These two properties together suggest that the large correction is due to the higher harmonics being amplified by the factor n^6. We examine this conjecture by decomposing the quadrupole moment into harmonics and determining the contribution of each. This is carried out by power series expansion in e in this Section (which seems at first to be a reasonable strategy since the total power is a finite polynomial in e), and independently by numerical integration in the next Section.

B. Fourier decomposition

Expand the quadrupole moment $q_{ij}(t)$ into a Fourier series

$$q_{ij}(t) \;=\; \sum_n \tilde{q}_{ij}(n, e)\, p(n\Omega t) \tag{5.1}$$

where for compact notation the periodic function $p(\theta)$ should be understood to be $\cos\theta$ for $(ij) = (xx), (yy), (zz)$ and $\sin\theta$ for $(ij) = (xy), (yx)$. We have also indicated explicitly that the expansion coefficients $\tilde{q}_{ij}(n, e)$ depend on e.

Since the final answer is to be compared to the reference case, the e-independent prefactors can be omitted and we write

$$q_{ij}(\phi) \;=\; \frac{h_{ij}(\phi)}{(1-e\cos\phi)^2} \tag{5.2}$$

With $n\Omega$ being the angular frequency of the nth harmonic, the power radiated goes like

$$\begin{aligned} P \;&=\; \sum_{ij} P_{ij} \\ P_{ij} \;&=\; (1-e^2)^4 \sum_n |\tilde{q}_{ij}(n,e)|^2\, n^6 \end{aligned} \tag{5.3}$$

The factor $(1-e^2)^4$ is due to the factor r_0^2 in q_{ij}; since $r_0 = a(1-e^2)$, r_0^4 will give $(1-e^2)^4$ and a^4 will be cancelled after comparing to the reference case. The factor n^6 in (5.3) is the essence of the conjecture.

C. Relation between angle and time

The analysis becomes somewhat cumbersome because q_{ij} is expressed in terms of ϕ, whereas we want to Fourier-analyze it in terms of t. The relation between t and ϕ is given in Appendix B. The point is that for a circular orbit ($e = 0$), t and ϕ are linearly related (the one advancing by a period T when the other advances by an angle 2π). The relationship becomes complicated for elliptic orbits, but can be analyzed order-by-order in powers of e. We write

$$\tilde{q}_{ij}(n,e) \;=\; \sum_{k=0}^{\infty} a_{ij}(n,k)\, e^k \tag{5.4}$$

The coefficients $a_{ij}(n,k)$ are evaluated up to $k = 8$ and shown in Table 1; they are nonzero only if $k \geq n-2$. (For example, the $n = 6$ harmonic starts with e^4.)

D. Checking total power

With all the coefficients in hand, we can compute P_{ij} (summed over harmonics) from (5.3), expressed in the form

$$P_{ij} \;=\; (1-e^2)^4 \sum_k b_{ij}(k)\, e^k \tag{5.5}$$

In detail, we have

$$\begin{aligned}
P_{xx} &= (1-e^2)^4\left(144+\frac{2957}{2}e^2+\frac{121825}{16}e^4+\frac{880065}{32}e^6+\frac{10176544}{128}e^8+\dots\right)\\
P_{xy} &= (1-e^2)^4\left(144+\frac{3033}{2}e^2+\frac{126927}{16}e^4+\frac{926955}{32}e^6+\frac{10805625}{128}e^8+\dots\right)\\
P_{yy} &= (1-e^2)^4\left(144+\frac{3113}{2}e^2+\frac{131377}{16}e^4+\frac{962505}{32}e^6+\frac{11233175}{128}e^8+\dots\right)\\
P_{zz} &= (1-e^2)^4\left(4e^2+31e^4+135e^6+\frac{3485}{8}e^8+\dots\right)
\end{aligned}\tag{5.6}$$

We note that these series, if terminated at some power of e, would still be positive-definite. (The truncation of a positive-definite infinite series need not be positive-definite; e.g., the function $(1-2e^2)^2 = 1-4e^2+4e^4$ when truncated to e^2 is not positive definite.)

These can be rendered into the form

$$P_{ij} \;=\; (1-e^2)^{-7/2}\sum_k c_{ij}(k)\, e^k \tag{5.7}$$

It is straightforward to compute the c coefficients from the b coefficients, and we have done so up to $k=8$. (a) It is verified that $c_{ij}(6)=0$, $c_{ij}(8)=0$, in agreement with the termination of the series in (4.9) — which provides a nice consistency check. (b) The coefficients $c_{ij}(0)$, $c_{ij}(2)$ and $c_{ij}(4)$ agree with (4.9). These results then confirm the validity of the harmonic analysis.

E. Separating into harmonics

It is then simple to isolate the contribution of each harmonic, by picking out only those terms with a given n. Define the power formula for each harmonic in analogy to (5.7), with coefficients $c_{ij}(n,k)$, of course satisfying

$$\sum_n c_{ij}(n,k) = c_{ij}(k) \tag{5.8}$$

and in particular

$$\sum_n c_{ij}(n,k) = 0 \quad \text{for } k>4 \tag{5.9}$$

The total power is then given by an analogous expression with coefficients $C(n,k)$, where

$$C(n,k) \;=\; \sum_{ij} c_{ij}(n,k) \tag{5.10}$$

satisfy

$$\sum_n C(n,k) \;=\; 0 \quad \text{for } k > 4 \tag{5.11}$$

The coefficients $C(n,k)$ are shown in Table 2, up to $k = 8$. These coefficients are *not* all positive.

These results in principle allows us to determine the contribution of each harmonic — but under one condition: that the number of terms evaluated (in our case up to e^8) is a sufficiently accurate approximation to the infinite series. This turns out not to be the case; in fact, the partial sum up to e^8 can be negative for some n. For example, if truncated at e^8, the $n = 5$ contribution is proportional to

$$C(5,6)e^6 + C(5,8)e^8 \;=\; 1.53 \times 10^5\, e^6 - 1.35 \times 10^6\, e^8$$

which is negative for $e^2 > 0.113$. Such negativity is nonsense and shows that we cannot truncation the sum over k.

This difficulty is somewhat surprising. Although the series in e^k for *each* harmonic does not converge well, the corresponding series in e^k for all harmonics taken together does converge well — in fact the series terminates. In other words, even though say $C(n,6)$ for each n is large, they actually add up to zero by a "miracle". This is readily verified from the entries in Table 2; for example

$$\sum_n C(n,6) \;=\; \frac{22921}{32} - 36756 + \frac{12594933}{64} - 313344 + \frac{9765625}{64} = 0 \tag{5.12}$$

This "miraculous" cancellation is the cause of the paradox: the expansion in e^k (to a moderate number of terms) does not work well for each harmonic, but does work well for their sum. Hence the method of expanding in powers e^k (unless carried to very large k) does not allow us to determine the contribution of each harmonic.

VI. HARMONIC ANALYSIS: NUMERICAL APPROACH

A. Formulation

Since the expansion in powers of e encounters an unexpected and subtle technical difficulty, in this Section we address the same issue using a numerical approach. Given the harmonic decomposition (5.1), the coefficients are given by

$$\tilde{q}_{ij}(n,e) \;=\; 2\int_0^1 q_{ij}(\phi)\, p(2n\pi\tau)\, d\tau \tag{6.1}$$

where $\tau = t/T$ and $p(\theta)$ is $\cos\theta$ ($\sin\theta$) if $q_{ij}(\phi)$ is even (odd) in ϕ.

The integral is to be carried out numerically. Using equal intervals of $\Delta\phi = 2\pi/\mathcal{N}$:

$$\tilde{q}_{ij}(n,e) \;\approx\; 2(\Delta\phi)\sum_k q_{ij}(\phi_k)\, p(2n\pi\tau_k)\left(\frac{d\tau}{d\phi}\right)_k \tag{6.2}$$

B. Relation between time t and angle ϕ

In order to carry out the evaluation, we need to relate ϕ and t. It will be convenient to consider the dimensionless variables $\tau = t/T$ and $\phi/2\pi$, each of which advances by one unit per cycle, and they are related by

$$\tau = \frac{t}{T} \;=\; \frac{\phi}{2\pi} + \Delta(e,\phi) \tag{6.3}$$

defining a periodic function Δ which vanishes at $\phi = 0, 2\pi$. (See Appendix B.)

From Kepler's second law, we can define a constant

$$J \;=\; r^2\frac{d\phi}{dt}$$

For elliptical orbit, it can be readily shown that

$$r \;=\; \frac{r_0}{1 - e\cos\phi}$$

Thus we have

$$J dt \;=\; \frac{r_0^2}{(1-e\cos\phi)^2}\, d\phi \tag{6.4}$$

or

$$T \;=\; \frac{r_0^2}{J}\, 2\pi g(e)$$

where

$$g(e) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{(1-e\cos\phi)^2} = (1-e^2)^{-\frac{3}{2}}$$

An expression for Δ is obtained by the indefinite integral of (6.4), after subtracting off the average value from both sides:

$$\Delta(e,\phi) \;=\; \frac{1}{2\pi}\int_0^{\phi} \Big[g(e)^{-1}(1-e\cos\phi)^{-2} - 1\Big]\, d\phi \tag{6.5}$$

This is evaluated numerically using the same step size $\Delta\phi = 2\pi/\mathcal{N}$. The result is shown in Figure 1 for various values of e,

C. Power

Consider the contribution of each harmonic to the power, i.e., each term in the sum (5.3):

$$P_{ij}(n) \;=\; (1-e^2)^4\, \tilde{q}_{ij}(n,e)^2\, n^6 \tag{6.6}$$

The power in each harmonic will further need to be summed over ij:

$$P(n) \;=\; P_{xx}(n) + 2P_{xy}(n) + P_{yy}(n) + P_{zz}(n) \tag{6.7}$$

and the total power is

$$P \;=\; \sum_n P(n) \tag{6.8}$$

which then allows us to assess the relative contributions of each harmonic.

The importance of high harmonics can be summarized by a mean value of n, defined as

$$\bar{n}(e) \;=\; \frac{\sum_n n\, P(n)}{\sum_n P(n)} \tag{6.9}$$

Test of convergence

It is necessary to test for convergence, i.e., to make sure that the results using finite $\mathcal{N}$ are accurate; we expect that the results for $P(n)$ would be inaccurate when $n/\mathcal{N} = O(1)$.

Figure 2 shows $\sum_{ij}\tilde{q}_{ij}(n,e)^2$ as bar charts, as function of n, computed for $e = 0.617$ using $\mathcal{N} = 72, 180$, i.e., $\Delta\phi = 5^{\mathrm{o}}, 2^{\mathrm{o}}$. The results can be summarized as follows.

- There is a region of relatively small n (in this case say up to $n \sim 10$) for which the two choices of discretization give consistent non-zero results, which are therefore reliable.
- There is a "valley" (in this case say $10 < n < 18$) for which these coefficients are nearly zero.

- Then for large n, typically $n/\mathcal{N} \sim 1/3$ or more, these computed coefficients are again non-zero, but disagreeing between the two choices of discretization.

It is obvious that the results in the third region are spurious, and all sum over n, e.g., as in (6.8) or (6.9), should be terminated in the "valley region".

For all further purposes, we use $\mathcal{N} = 3600$, i.e., $\Delta\phi = 0.1^o$.

Comparison with analytic results

Table 3 provides a comparison between the analytical results in Section IV and the numerical result by summing over harmonics as in (6.8), for a range of eccentricities. Also shown is the value of n at which the summation was truncated for each eccentricity. The numerical results for the total power agree well with the analytic results, giving further confidence to the separation into harmonics.

D. Results

Here, we present results for the case $e = 0.617$. The normalized power spectrum, computed up to the $n = 20$, is shown in Figure 3. Apart from the normalization, Figure 3 differs from Figure 2 in that it shows the power, which contains the extra factor n^6. In fact, the contrast between these two figures illustrates clearly the role of this factor.

The average n, as defined in (6.9), is $\bar{n}(0.617) = 8.3$.

Figure 4, shows $\bar{n}(e)$ as a function of e. This increases rapidly with e, especially beyond say $e = 0.5$. This feature is the final result of this Section, and vividly confirms the conjecture that the large eccentricity factor can be attributed to the higher harmonics which enter with we weights n^6.

VII. CONCLUSION

We have evaluated, through an expansion in powers of e, the power radiated by a binary system, and the result agrees well with data from the Hulse–Taylor pulsar. The calculation can be expressed as the sum over different harmonics. Each of these contributions is *not* well represented by a low-order truncation of the power-series expansion in e — even though their sum is well represented. The contribution of each harmonic can nevertheless be evaluated numerically. It is confirmed that as e increases, the higher harmonics become important, and this feature is nicely summarized by the behavior of $\bar{n}(e)$ in Figure 4.

In short, we have understood both the value of the eccentricity factor and the physical reason why it is so large in the case of the Hulse–Taylor pulsar.

DECLARATION OF CONTRIBUTIONS

The work up to Section V was carried out by Lee over the course of more than a year, up to summer 2017, as an undergraduate project. Lai joined the project as a summer visiting student to CUHK and carried out the work in Section VI. It is understood and agreed that each co-author, to satisfy course requirements at CUHK and UCLA respectively, may prepare and submit single-authored reports, provided (a) there is a clear statement about the individual contributions, (b) the focus is on the single author's own work with the rest providing only a background, (c) reference is made to this joint report and (d) acknowledgement is given to the collaborator.

ACKNOWLEDGEMENT

This project is supervised by Prof K Young of the Department of Physics, The Chinese University of Hong Kong. William Mccorkindale (Cambridge) contributed to some of the earlier work.

APPENDIX A: TWO WAYS OF EXPRESSING THE RATE OF ORBIT DECAY

The definition of α is given by

$$\alpha \;=\; -\frac{\dot{T}}{T} \tag{A1}$$

The currently accepted value is $\alpha = 2.72 \times 10^{-9}\,\mathrm{yr}^{-1}$. For $\alpha t \ll 1$

$$T \;=\; T_0 e^{-\alpha t} \approx T_0(1 - \alpha t) \tag{A2}$$

Hence for each cycle, there will be a time difference of αT_0^2. In the nth cycle, the time difference is $n\alpha T_0^2$. The cumulative time difference after N cycles is

$$\Delta t_c \;=\; \sum_{n=1}^{N} n\alpha T_0^2 \approx \frac{N^2}{2}\alpha T_0^2 \tag{A3}$$

But $N = t/T_0$, hence

$$\Delta t_c \; = \; \frac{\alpha}{2} t^2 \tag{A4}$$

The value of $\alpha/2$ is 4.3 s/decade^2.

APPENDIX B: SOME PROPERTIES OF KEPLER ORBITS

By Kepler's Second Law,

$$J dt \; = \; r^2 d\phi = \frac{r_0^2}{(1 - e\cos\phi)^2}\, d\phi$$

where J is a constant. Integrating over a period:

$$T \; = \; \frac{r_0^2}{J} \int_0^{2\pi} \frac{d\phi}{(1 - e\cos\phi)^2} = \frac{r_0^2}{J} \cdot 2\pi g(e)$$

where

$$g(e) \; = \; \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{(1 - e\cos\phi)^2}$$

The exact solution of $g(e)$ is [11]

$$g(e) \; = \; (1 - e^2)^{-3/2} \tag{B1}$$

Thus

$$T \; = \; 2\pi \frac{r_0^2}{J} (1 - e^2)^{-3/2} \tag{B2}$$

To find the relationship between time and angle, we go back to (B1) and write it as

$$\begin{aligned} dt \; &= \; \frac{r_0^2}{J(1 - e\cos\phi)^2}\, d\phi \\ &= \; \frac{r_0^2}{J} \left\{ g(e) + \left[\frac{1}{(1 - e\cos\phi)^2} - g(e) \right] \right\} d\phi \end{aligned}$$

We have added and subtracted a term $g(e)$ in the integrand so that the square bracket has zero average. Thus

$$\begin{aligned} t \; &= \; \frac{r_0^2 g(e)}{J} \phi + \frac{{r_0}^2}{J} \int_0^{\phi} \left[\frac{1}{(1 - e\cos\phi)^2} - g(e) \right] d\phi \\ &= \; T\frac{\phi}{2\pi} + \frac{r_0^2}{J} \int_0^{\phi} \left[\frac{1}{(1 - e\cos\phi)^2} - g(e) \right] d\phi \end{aligned}$$

where the first term on the RHS has been identified in terms of the period T by using the property that the second term vanishes for $\phi = 2\pi$. Hence

$$\frac{t}{T} \;=\; \frac{\phi}{2\pi} + \Delta(e,\phi) \tag{B3}$$

where

$$\Delta \;=\; \frac{1}{2\pi}\int_0^{\phi}\left[g(e)^{-1}(1-e\cos\phi)^{-2}-1\right]d\phi$$

Note that in (B3), on the RHS the first term captures the secular dependence and Δ is strictly periodic: $\Delta(e,2\pi)=0$; this is the reason for adding and subtracting $g(e)$.

APPENDIX C: EVALUATION OF THIRD TIME DERIVATIVE

Recall that

$$P_{ij} \;=\; \frac{J^5}{r_0^6 T}\int_0^{2\pi}\left[\left(s^2\frac{d}{d\phi}\right)^3\frac{h_{ij}}{s^2}\right]^2\frac{1}{s^2}\,d\phi$$

where h_{ij} is a polynomial in $\cos\phi$ and/or $\sin\phi$. We evaluate P_{ij} term by term. For the purpose of this Appendix, it is convenient to adopt the compact abbreviation

$$C \equiv \cos\phi \;\;,\;\; S \equiv \sin\phi$$

The xx term

For the term P_{xx}

$$P_{xx} \;=\; \frac{J^5}{r_0^6 T}\int_0^{2\pi}\left[\left(s^2\frac{d}{d\phi}\right)^3\left(\frac{3C^2-1}{s^2}\right)\right]^2\frac{1}{s^2}\,d\phi$$

First differentiation

$$(1-eC)^2\frac{d}{d\phi}\frac{3C^2-1}{(1-eC)^2}$$
$$=\;\frac{S(2e-6C)}{1-eC}$$

Note that one factor of $(1-eC)$ in the denominator has been removed.

Second differentiation

$$(1-eC)^2\frac{d}{d\phi}\frac{S(2e-6C)}{1-eC}$$
$$=\;6eC^3-12C^2+2eC+6-2e^2$$

Note that another factor of $(1 - eC)$ in the denominator has been removed, and we are left with a polynomial in e and C, S.

<u>Third differentiation</u>

$$\begin{aligned} & (1-eC)^2 \frac{d}{d\phi}(6eC^3 - 12C^2 + 2eC + 6 - 2e^2) \\ = & -2S(9eC^2 - 12C + e)(1-eC)^2 \end{aligned}$$

The property that the integrand ends up as such a polynomial, importantly without trigonometric functions in the denominator, is common to all the terms and we shall not repeat these remarks below.

The integral (ignoring the constants in the prefactor) becomes

$$\begin{aligned} & \int_0^{2\pi} 4S^2(9eC^2 - 12C + e)^2(1-eC)^2 \, d\phi \\ = & \left(144 + \frac{797}{2}e^2 + \frac{565}{16}e^4\right)\pi \end{aligned}$$

The power P_{xx} is then

$$P_{xx} = \frac{\pi J^5}{r_0^6 T}\left(144 + \frac{797}{2}e^2 + \frac{565}{16}e^4\right)$$

The xy and yx terms

For the term P_{xy}

$$P_{xy} = \frac{J^5}{r_0^6 T}\int_0^{2\pi}\left[\left(s^2\frac{d}{d\phi}\right)^3\left(\frac{3SC}{s^2}\right)\right]^2 \frac{1}{s^2}\, d\phi$$

<u>First differentiation</u>

$$\begin{aligned} & (1-eC)^2 \frac{d}{d\phi}\frac{3SC}{(1-eC)^2} \\ = & \frac{6C^2 - 3eC - 3}{1-eC} \end{aligned}$$

<u>Second differentiation</u>

$$\begin{aligned} & (1-eC)^2 \frac{d}{d\phi}\frac{6C^2 - 3eC - 3}{1-eC} \\ = & 6S(eC^2 - 2C + e) \end{aligned}$$

Third differentiation

$$
\begin{aligned}
& (1-eC)^2 \frac{d}{d\phi} 6S(eC^2 - 2C + e) \\
= \; & 6(3eC^3 - 4C^2 - eC + 2)(1-eC)^2
\end{aligned}
$$

The integral becomes

$$
\begin{aligned}
& \int_0^{2\pi} 36(3eC^3 - 4C^2 - eC + 2)^2 (1-eC)^2 \, d\phi \\
= \; & \left(144 + \frac{873}{2}e^2 + \frac{1107}{16}e^4\right)\pi
\end{aligned}
$$

The power P_{xy} is then

$$
P_{xy} \; = \; \frac{\pi J^5}{r_0^6 T}\left(144 + \frac{873}{2}e^2 + \frac{1107}{16}e^4\right)
$$

Note that the expression for $h_{ij}(\phi)$ is identical for P_{xy} and P_{yx}, so $P_{xy} = P_{yx}$.

The yy term

For the term P_{yy}

$$
P_{yy} \; = \; \frac{J^5}{r_0^6 T} \int_0^{2\pi} \left[\left(s^2 \frac{d}{d\phi} \right)^3 \left(\frac{3S^2 - 1}{s^2} \right) \right]^2 \frac{1}{s^2} \, d\phi
$$

First differentiation

$$
\begin{aligned}
& (1-eC)^2 \frac{d}{d\phi} \frac{3S^2 - 2}{(1-eC)^2} \\
= \; & \frac{2S(3C - 2e)}{1 - eC}
\end{aligned}
$$

Second differentiation

$$
\begin{aligned}
& (1-eC)^2 \frac{d}{d\phi} \frac{2S(3C - 2e)}{1 - eC} \\
= \; & -6eC^3 + 12C^2 - 4eC + 4e^2 - 6
\end{aligned}
$$

Third differentiation

$$
\begin{aligned}
& (1-eC)^2 \frac{d}{d\phi} (-6eC^3 + 12C^2 - 4eC + 4e^2 - 6) \\
= \; & 2S(9eC^2 - 12C + 2e)(1-eC)^2
\end{aligned}
$$

The integral becomes

$$\int_0^{2\pi} 4S^2(9eC^2 - 12C + 2e)^2(1-eC)^2\,d\phi$$
$$= \left(144 + \frac{953}{2}e^2 + \frac{757}{16}e^4\right)\pi$$

The power P_{yy} is then

$$P_{yy} = \frac{\pi J^5}{r_0^6 T}\left(144 + \frac{953}{2}e^2 + \frac{757}{16}e^4\right)$$

The zz term

For the term P_{zz}

$$P_{zz} = \frac{J^5}{r_0^6 T}\int_0^{2\pi}\left[\left(s^2\frac{d}{d\phi}\right)^3\left(\frac{-1}{s^2}\right)\right]^2\frac{1}{s^2}\,d\phi$$

First differentiation

$$(1-eC)^2\frac{d}{d\phi}\frac{-1}{(1-eC)^2}$$
$$= \frac{2eS}{1-eC}$$

Second differentiation

$$(1-eC)^2\frac{d}{d\phi}\frac{2eS}{1-eC}$$
$$= 2e(C-e)$$

Third differentiation

$$(1-eC)^2\frac{d}{d\phi}[2e(C-e)]$$
$$= -2eS(1-eC)^2$$

The integral becomes

$$\int_0^{2\pi} 4e^2S^2(1-eC)^2\,d\phi = (4e^2+e^4)\pi$$

The power P_{zz} is then

$$P_{zz} = \frac{\pi J^5}{r_0^6 T}(4e^2+e^4) \tag{C1}$$

APPENDIX D: TOY MODEL FOR PARADOX

The technical difficulty encountered can be stated abstractly as follows. There is a function

$$F(e) \;=\; \sum_n f(n,e) \tag{D1}$$

Think of n as the harmonic index. All these functions can be expanded in powers of e^2:

$$\begin{aligned} F(e) \;&=\; \sum_k C(k)\, e^{2k} \\ f(n,e) \;&=\; \sum_k c(n,k)\, e^{2k} \end{aligned} \tag{D2}$$

with the property that the series for F terminates but that for $f(n,e)$ does not, and perhaps converges slowly or even not at all. In other words, for $k > k_0$ for some k_0, there is the "miracle"

$$\sum_n c(n,k) \;=\; 0 \tag{D3}$$

How is this possible? Here we construct a toy model that exhibits this feature.

Start with the identity

$$(1-1)^{k+m}\; e^{2k} \;=\; 0 \tag{D4}$$

where $m > 0$ is an integer. Expand the bracket:

$$\sum_n (-1)^n \frac{(k+m)!}{(k+m-n)!\, n!}\, e^{2k} \;=\; 0 \tag{D5}$$

Now multiply by A_k, sum over k and call the result $F(e)$:

$$F(e) \;=\; \sum_k A_k \cdot \sum_n (-1)^n \frac{(k+m)!}{(k+m-n)!\, n!}\, e^{2k} = 0 \tag{D6}$$

Now isolate the terms associated with n:

$$f(n,e) \;=\; (-1)^n \sum_k A_k \frac{(k+m)!}{(k+m-n)!\, n!}\, e^{2k} \tag{D7}$$

Thus we identify the coefficients as

$$c(n,k) \;=\; (-1)^n A_k \frac{(k+m)!}{(k+m-n)!\, n!} \tag{D8}$$

We see that the sum (D7) does not terminate, and depending on the choice of A_k can converge very slowly or even not converge at all.

If we change $c(n,k)$ for $k \leq k_0$, we would get $F(e)$ which is a polynomial but not zero.

---

[1] B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), "Observation of gravitational waves from a binary black hole merger", Phys. Rev. Lett. **116**, 061102 (2016).

[2] B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), "GW151226: Observation of gravitational waves from a 22-solar-mass binary black hole coalescence", Phys. Rev. Lett. **116**, 241103 (2016).

[3] J.H. Taylor and J.M. Weisberg, "A new test of general relativity: Gravitational radiation and the binary pulsar PSR 1913+16", Ap. J. **253**, 908 (1982).

[4] J.M. Weisberg, D.J. Nice and J.H. Taylor, "Timing measurements of the relativistic binary pulsar PSR B1913+16", Ap. J. **722**, 1030 (2010).

[5] The Nobel Prize Foundation, "The Nobel Prize in Physics" (1993).

[6] Thus $2a_1$ is about twice as large as its projection along the line of sight.

[7] K. Young, *Lecture notes for classical electrodynamics*, Chapter 2: "Electrostatics" (2016).

[8] K. Young, *Lecture notes for classical electrodynamics*, Chapter 10: "Radiation by a harmonic source: some applications" (2016).

[9] J.D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1999).

[10] In this Section, it should be clear that indices attached to r denote particle number. To avoid confusion with a power, the indices 1 and 2 are written as subscripts.

[11] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Trans. A. Jeffrey, Acad. Press, 1965)

[12] T. Padmanabhan, *Gravitation: Foundations and Frontiers* (Cambridge University Press, New York, 2010)

Tables

n	k	$a_{xx}(n,k)$	$a_{xy}(n,k)$	$a_{yy}(n,k)$	$a_{zz}(n,k)$
0	0	1/2	-	1/2	−1
0	2	11/2	-	−2	−7/2
0	4	21/2	-	−9/2	−6
0	6	31/2	-	−7	−17/2
0	8	41/2	-	−19/2	−11
1	1	11/2	3/2	−7/2	−2
1	3	71/8	−3/4	−41/8	−15/4
1	5	9523/768	1927/256	−5291/768	−529/96
1	7	733597/46080	44607/5120	−398567/46080	−33503/4608
2	0	3/2	3/2	−3/2	0
2	2	−1	−3/4	1/2	1/2
2	4	−65/48	−3/4	25/48	5/6
2	6	−251/120	−253/240	217/240	19/16
2	8	−16103/5760	−1639/1280	7231/5760	1109/720
3	1	−3/2	−3/2	3/2	0
3	3	11/16	9/16	−7/16	−1/4
3	5	431/1280	99/1280	29/1280	−23/64
3	7	1827/2560	363/1280	−273/1280	−1281/2560
4	2	3/2	3/2	−3/2	0
4	4	−5/6	−3/4	2/3	1/6
4	6	1/30	43/240	−7/30	1/5
4	8	−221/630	−129/1120	23/315	5/18
5	3	−25/16	−25/16	25/16	0
5	5	875/768	275/256	−775/768	−25/192
5	7	−10475/32256	−375/896	3625/8064	−575/4608
6	4	27/16	27/16	−27/16	0
6	6	−63/40	−243/160	117/80	9/80
6	8	6147/8960	6723/8960	−981/1280	9/112
7	5	−2401/1280	−2401/1280	2401/1280	0
7	7	199283/92160	21609/10240	−189679/92160	−2401/23040
8	6	32/15	32/15	−32/15	0
8	8	−928/315	−304/105	128/45	32/315
9	7	−177147/71680	−177147/71680	177147/71680	0
10	8	15625/5376	15625/5376	−15625/5376	0

Table 1. The coefficients $a_{ij}(nk)$. All combinations of n and k that are not shown are above yield the value 0 for the coefficients.

n	k	$C(n,k)$
1	2	87
1	4	$-1575/4$
1	6	$22921/32$
1	8	$-510613/768$
2	0	576
2	2	-4896
2	4	17880
2	6	-36756
2	8	$281419/6$
3	2	6561
3	4	$-216513/4$
3	6	$12594933/64$
3	8	$-212771043/512$
4	4	36864
4	6	-313344
4	8	$3576320/3$
5	6	$9765625/64$
5	8	$-693359375/512$
6	8	531441

Table 2. The coefficients $C(n,k)$. Note that $C(n,k)=0$ whenever k is odd.

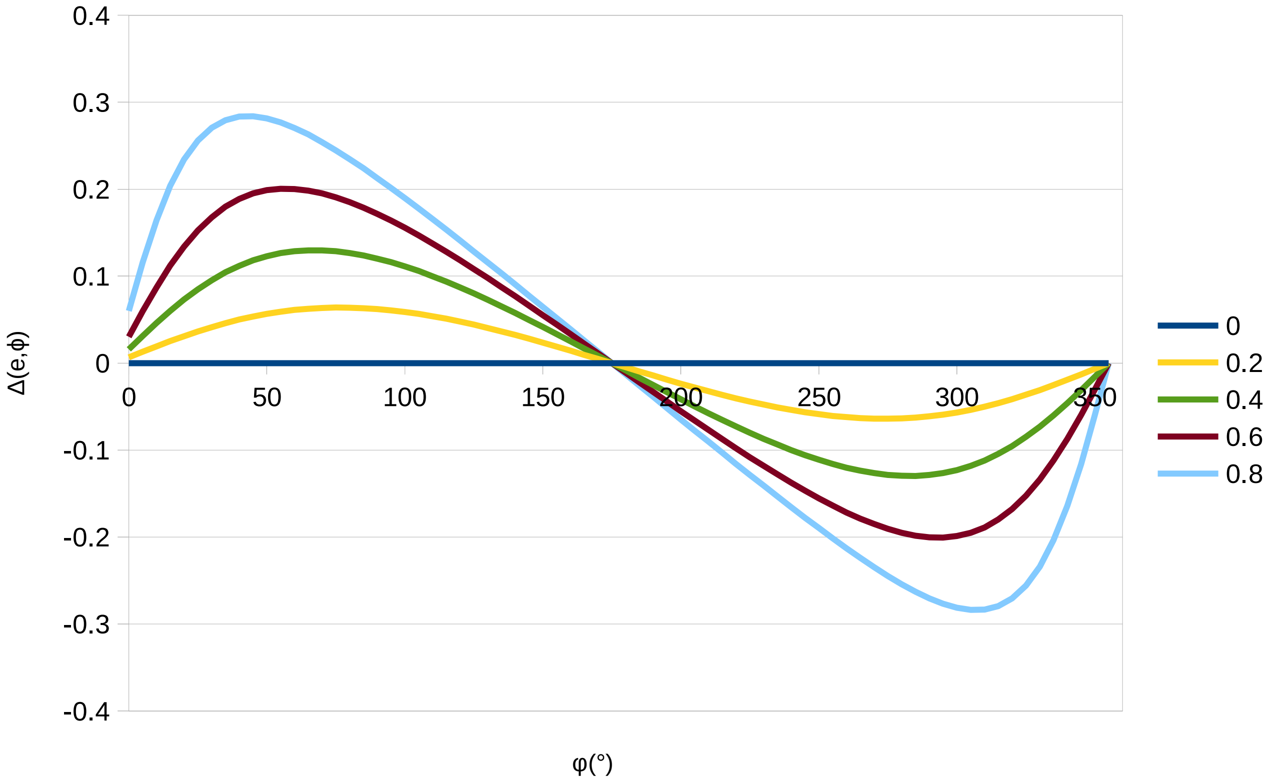

Figure 1: The periodic function $\Delta(e,\phi)$, evaluated for several different values of e.

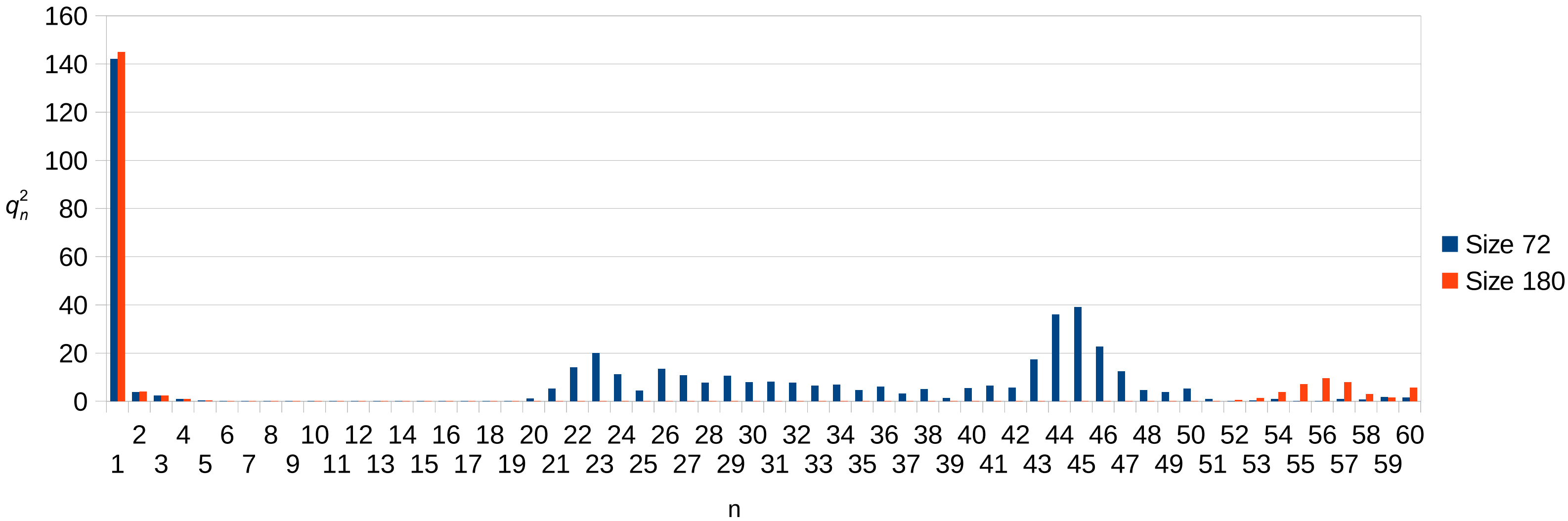

Figure 2: The computed value of q_n^2 (summed over polarizations) versus *n*. The blue (red) bars are computed using 72 (180) integration steps. For small *n* (in this case, up to 20), the results have converged, whereas for large *n* (scaling with number of integration steps), the numbers are spurious.

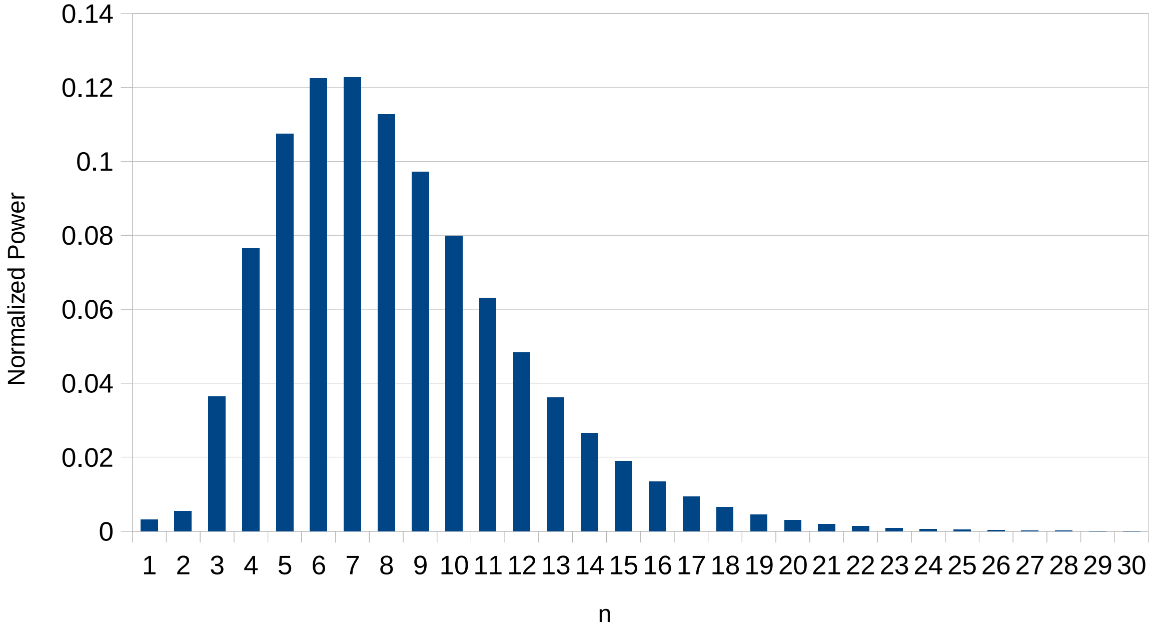

Figure 3: The power in each harmonic (normalized to the total power), versus *n*, for the case of *e* = 0.617; evaluated using 3600 steps. This histogram differs from the one used in Figure 2 by the factor n^6.

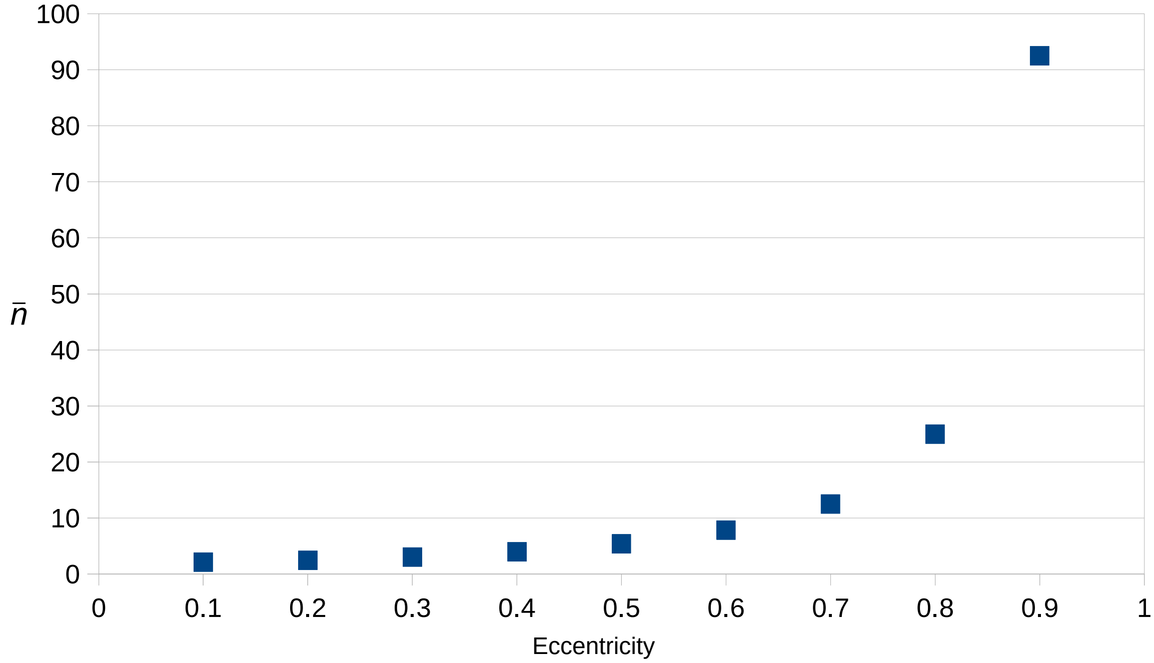

Figure 4: The mean $\bar{n}$ versus e. The conjecture is confirmed by the large values.